Math 4 Honors
Lesson 3-4: *Partial Fraction Decomposition*

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Date ______________________

Learning Goal:

- *I can find the partial fraction decomposition of a rational expression.*

In Calculus, there are several procedures that are much easier if we can take a large fraction and break it up into pieces. The procedure that can decompose larger fractions is called Partial Fraction Decomposition. We will proceed as if we are working backwards through an addition of fractions with LCD.

Example 1: For our first example we will work an LCD problem frontwards and backwards. Use an LCD to complete the following addition.

LCD: $(x+2)(x-1)$

$$\frac{3}{x+2}+\frac{5}{x-1}=\frac{3(x-1)}{(x+2)(x-1)}+\frac{5(x+2)}{(x+2)(x-1)}=\frac{3x-3+5x+10}{(x+2)(x-1)}$$

$$=\frac{8x+7}{(x+2)(x-1)}$$

Now let's work this problem backwards . . . .

Find the *partial fraction decomposition* for $\frac{8x+7}{x^2+x-2}$.

As we saw ~~in the previous slide~~ above that the denominator factors as $(x+2)(x-1)$. We want to find numbers A and B so that:

$$\frac{8x+7}{x^2+x-2}=\frac{A}{x+2}+\frac{B}{x-1}$$

Study the process below and describe what was done in each step of the process.

$$\underbrace{(x+2)(x-1)}_{LCD}\left(\frac{8x+7}{x^2+x-2}\right)=\left(\frac{A}{x+2}+\frac{B}{x-1}\right)\underbrace{(x+2)(x-1)}_{LCD}$$ Multiply both sides by LCD

$$\cancel{(x+2)}\cancel{(x-1)}\left(\frac{8x+7}{\cancel{(x+2)}\cancel{(x-1)}}\right)=\frac{A}{\cancel{x+2}}\cancel{(x+2)}(x-1)+\frac{B}{\cancel{x-1}}(x+2)\cancel{(x-1)}$$ Distribute LCD on right side; Divide out common factors

$8x+7=A(x-1)+B(x+2)$ Distribute A & B on right side

$8x+7=Ax-A+Bx+2B$ Combine like terms + factor

$8x+7=(A+B)x+(-A+2B)$

If the left side and the right side are going to be equal then:

$A+B=8$ and $-A+2B=7$. Why? A+B is coeff. of x & so is 8 / -A+2B & 7 are constants

2 equations; 2 variables; time to solve a system! Show your work below:

$$\begin{aligned} \cancel{A}+B&=8\\ -\cancel{A}+2B&=7\\ \hline 3B&=15\\ B&=5 \end{aligned}$$

$A+5=8$

$A=3$

What is significant about the solution to the system? Take a look at what we started with!

$$\frac{8x+7}{x^2+x-2}=\frac{A}{x+2}+\frac{B}{x-1}=\frac{3}{x+2}+\frac{5}{x-1}$$

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Example 2: The previous example had an LCD comprised of two linear factors. This next example has a denominator with one linear factor and one quadratic factor.

Find the partial fraction decomposition for $\dfrac{-x^2+11x-10}{x^3+3x^2+4x+12}$

First factor the denominator. Use the factor by grouping technique:

$(x^3+3x^2)+(4x+12) = x^2(x+3)+4(x+3)$

$= (x+3)(x^2+4)$

So, $\dfrac{-x^2+11x-10}{x^3+3x^2+4x+12} = \dfrac{-x^2+11x-10}{(x^2+4)(x+3)} = \dfrac{Ax+B}{x^2+4}+\dfrac{C}{x+3}$

Because one of the factors in the denominator is quadratic, it is quite possible that its numerator could have an x term and a constant term—thus the use of $Ax+B$ in the numerator.

Describe what was done in each step:

$\left(\cancel{(x^2+4)}\cancel{(x+3)}\right)\left(\dfrac{-x^2+11x-10}{\cancel{(x^2+4)}\cancel{(x+3)}}\right) = \left(\dfrac{Ax+B}{x^2+4}+\dfrac{C}{x+3}\right)\left((x^2+4)(x+3)\right)$ Multiply both sides by LCD

$-x^2+11x-10 = \dfrac{Ax+B}{\cancel{x^2+4}}\left(\cancel{(x^2+4)}(x+3)\right)+\dfrac{C}{\cancel{x+3}}\left((x^2+4)\cancel{(x+3)}\right)$ Multiply & simplify

$-x^2+11x-10 = Ax^2+3Ax+Bx+3B+Cx^2+4C$ Distribute on right side

$-x^2+11x-10 = (A+C)x^2+(3A+B)x+(3B+4C)$ Combine like terms & factor on right side

Now there are three variables, so it is necessary to set up a system of three equations. Write the equations below and solve the system.

$-1 = A+C \qquad 11 = 3A+B \qquad -10 = 3B+4C$

$$\begin{aligned} A + \quad\; C &= -1 \\ 3A + B \quad\; &= 11 \\ 3B+4C &= -10 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 0 \\ 0 & 3 & 4 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} -1 \\ 11 \\ -10 \end{bmatrix}$$

Matrix time!

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$$

Use your solution to complete the partial fractions.

$$\frac{-x^2+11x-10}{x^3+3x^2+4x+12} = \frac{\boxed{3}x+\boxed{2}}{x^2+4}+\frac{\boxed{-4}}{x+3}$$

➢ How can you check verify that you have found the correct partial fractions?

Add these up!

Call The Heinl over for verification of your work and solutions. *Heinl Approval:* $\boxed{\text{JH}}$

Example 3: Let's see what happens when one of the factors in the denominator is raised to a power. Consider the following for partial fraction decomposition:

$$\frac{13x^2+48x+72}{x^3+6x^2+9\cancel{x}}=\frac{13x^2+48x+72}{x\left(x^2+6x+9\right)}=\frac{13x^2+48x+72}{x\left(x+3\right)^2}$$

The following set up considers that the second fraction could have come from two pieces.

$$\frac{13x^2+48x+72}{x\left(x+3\right)^2}=\frac{A}{x}+\frac{B}{x+3}+\frac{C}{(x+3)^2}$$

Describe what was done in each step:

$$\left(x(x+3)^2\right)\left(\frac{13x^2+48x+72}{x(x+3)^2}\right)=\left(\frac{A}{x}+\frac{B}{x+3}+\frac{C}{(x+3)^2}\right)\left(x(x+3)^2\right)$$ Multiply both sides by LCD

$$13x^2+48x+72=\frac{A}{\cancel{x}}\cancel{x}(x+3)^2+\frac{B}{\cancel{x+3}}x(x+3)^{\cancel{2}}+\frac{C}{\cancel{(x+3)^2}}x\cancel{(x+3)^2}$$ Distribute LCD & simplify

$$13x^2+48x+72=A(x+3)^2+Bx(x+3)+Cx$$ Divide out common factors

$$13x^2+48x+72=A\left(x^2+6x+9\right)+Bx^2+3Bx+Cx$$ Expand $(x+3)^2$

$$13x^2+48x+72=Ax^2+6Ax+9A+Bx^2+3Bx+Cx$$ Distribute on right side

$$13x^2+48x+72=(A+B)x^2+(6A+3B+C)x+9A$$ Combine like terms

Now there are three variables, so it is necessary to set up a system of three equations. Write the equations below and solve the system.

$A+B=13$ $\quad 6A+3B+C=48$ $\quad 72=9A$

$$\begin{aligned} A+B &= 13\\ 6A+3B+C &= 48\\ 9A &\quad 72\end{aligned}$$

$$\begin{bmatrix}1&1&0\\6&3&1\\9&0&0\end{bmatrix}\cdot\begin{bmatrix}A\\B\\C\end{bmatrix}=\begin{bmatrix}13\\48\\72\end{bmatrix}$$

Matrix time!

$$\begin{bmatrix}A\\B\\C\end{bmatrix}=\begin{bmatrix}8\\5\\-15\end{bmatrix}$$

Use your solution to complete the partial fractions.

$$\frac{13x^2+48x+72}{x^3+6x^2+9}=\frac{8}{x}+\frac{5}{x+3}-\frac{15}{(x+3)^2}$$

Call The Heinl over for verification of your work and solutions. *Heinl Approval:* $\boxed{\text{JH}}$

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Example 4: Find the partial fraction decomposition for $\frac{2x^3-4x^2-15x+5}{x^2-2x-8}$.

Notice how this is an improper fraction. Before starting the process, we must divide the numerator by the denominator and then decompose the remainder.

$$\begin{array}{r} 2x + \boxed{\frac{x+5}{x^2-2x-8}}\ r(x) \\ x^2-2x-8\overline{)\,2x^3-4x^2-15x+5} \\ \underline{2x^3-4x^2-16x} \\ x+5 \end{array}$$

So, $\frac{2x^3-4x^2-15x+5}{x^2-2x-8} = 2x + \boxed{\frac{x+5}{x^2-2x-8}}$ r(x)   Denom: $(x-4)(x+2)$

Now it's your turn. Decompose the remainder.

$$(x-4)(x+2)\left[\frac{x+5}{(x-4)(x+2)}\right] = \left[\frac{A}{x-4} + \frac{B}{x+2}\right](x-4)(x+2)$$

$$x+5 = \left(\frac{A}{\cancel{x-4}}\right)\cancel{(x-4)}(x+2) + \left(\frac{B}{\cancel{x+2}}\right)(x-4)\cancel{(x+2)}$$

$$x+5 = A(x+2) + B(x-4)$$
$$x+5 = Ax + 2A + Bx - 4B$$
$$x+5 = Ax + Bx + 2A - 4B$$
$$x+5 = x(A+B) + 2A - 4B$$

$$-2(A+B=1)$$
$$2A-4B=5$$
$$-2A-2B=-2$$
$$-6B=3$$
$$B=-\frac{1}{2}$$
$$A-\frac{1}{2}=1$$
$$A=\frac{3}{2}$$

Complete: $\frac{2x^3-4x^2-15x+5}{x^2-2x-8} = 2x + \frac{3}{2(x-4)} - \frac{1}{2(x+2)}$

Call The Heinl over for verification of your work and solutions. *Heinl Approval:* $\boxed{\text{2H}}$

Summarize the process of *Partial Fraction Decomposition:* * (Divide 1st if improper fraction)

1. Factor denominator
2. Set up your fractions w/ A, B, C, etc in num.
3. Multiply both sides of equation by LCD
4. Distribute / Simplify / combine like terms
5. Set up & solve system to find A, B, C values
6. Write your final Decomposed equation

Symbolab Partial Fractions Calculator

Find the partial fractions of a fractions step by step

https://www.symbolab.com/solver/partial-fractions-calculator

$$\text{partial fractions } \frac{13x^2+48x+72}{x^3+6x^2+9x}$$

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Examples »

Solution

« Hide Steps

Take the partial fraction of $\frac{13x^2+48x+72}{x^3+6x^2+9x}$: $\frac{5}{x+3}-\frac{15}{(x+3)^2}+\frac{8}{x}$

Steps

$$\frac{13x^2+48x+72}{x^3+6x^2+9x}$$

Factor x^3+6x^2+9x: $x(x+3)^2$ Show Steps

$$=\frac{13x^2+48x+72}{x(x+3)^2}$$

Create the partial fraction template using the denominator $x(x+3)^2$ Show Steps

$$\frac{13x^2+48x+72}{x(x+3)^2}=\frac{a_0}{x}+\frac{a_1}{x+3}+\frac{a_2}{(x+3)^2}$$

Multiply equation by the denominator

$$\frac{x(x+3)^2(13x^2+48x+72)}{x(x+3)^2}=\frac{a_0x(x+3)^2}{x}+\frac{a_1x(x+3)^2}{x+3}+\frac{a_2x(x+3)^2}{(x+3)^2}$$

Simplify

$$13x^2+48x+72=a_2x+a_1x(x+3)+a_0(x+3)^2$$

Solve the unknown parameters by plugging the real roots of the denominator: $0, \ -3$

For the denominator root 0: $a_0=8$ Show Steps

For the denominator root -3: $a_2=-15$ Show Steps

$$a_0=8, a_2=-15$$

Plug in the solutions to the known parameters

$$13x^2+48x+72=a_1x(x+3)+(-15)x+8(x+3)^2$$

Expand

$$13x^2+48x+72=8x^2+a_1x^2+3a_1x+33x+72$$

Group elements according to powers of x

$$13x^2+48x+72=x^2(a_1+8)+x(3a_1+33)+72$$

Solve $3a_1+33=48$ for a_1

$$a_1=5$$

Plug the solutions to the partial fraction parameters to obtain the final result

$$\frac{(-15)}{(x+3)^2}+\frac{8}{x}+\frac{5}{x+3}$$

Simplify

$$\frac{5}{x+3}-\frac{15}{(x+3)^2}+\frac{8}{x}$$

Save

Lesson 3-4 Homework

In Exercises 5–8, match the rational expression with the form of its decomposition. [The decompositions are labeled (a), (b), (c), and (d).]

(a) $\frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$ (b) $\frac{A}{x} + \frac{B}{x-4}$

(c) $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-4}$ (d) $\frac{A}{x} + \frac{Bx+C}{x^2+4}$

5. $\frac{3x-1}{x(x-4)}$ B
6. $\frac{3x-1}{x^2(x-4)}$ C
7. $\frac{3x-1}{x(x^2+4)}$ D
8. $\frac{3x-1}{x(x^2-4)}$ A $(x+2)(x-2)$

Show all work on another sheet of paper.

Write the partial fraction decomposition of each rational expression.

21. $\frac{1}{2x^2+x} = \frac{1}{x(2x+1)}$

23. $\frac{3}{x^2+x-2} = \frac{3}{(x+2)(x-1)}$

27. $\frac{x^2+12x+12}{x^3-4x} = \frac{x^2+12x+12}{x(x^2-4)} = \frac{x^2+12x+12}{x(x+2)(x-2)}$

29. $\frac{3x}{(x-3)^2}$

31. $\frac{4x^2+2x-1}{x^2(x+1)}$

35. $\frac{x}{(x^3-x^2)-(2x+2)} = \frac{x}{x^2(x-1)-2(x-1)} = \frac{x}{(x^2-2)(x-1)}$

37. $\frac{2x^2+x+8}{(x^2+4)^2}$

39. $\frac{x}{16x^4-1} = \frac{x}{(4x^2+1)(4x^2-1)} = \frac{x}{(4x^2+1)(2x-1)(2x+1)}$

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45. $\dfrac{2x^3 - x^2 + x + 5}{x^2 + 3x + 2}$

$$\begin{array}{r} 2x - 7 \\ x^2+3x+2 \overline{)\,2x^3 - x^2 + x + 5} \\ -(2x^3 + 6x^2 + 4x) \\ \hline -7x^2 - 3x + 5 \\ -(-7x^2 - 21x - 14) \\ \hline 18x + 19 \end{array}$$

47. $\dfrac{x^4}{(x-1)^3}$

$$\begin{array}{r} x + 3 \\ x^3 - 3x^2 + 3x - 1 \overline{)\,x^4} \\ -(x^4 - 3x^3 + 3x^2 - x) \\ \hline 3x^3 - 3x^2 + x \\ -(3x^3 - 9x^2 + 9x - 3) \\ \hline 6x^2 - 8x + 3 \end{array}$$

49. $\dfrac{x^4 + 2x^3 + 4x^2 + 8x + 2}{x^3 + 2x^2 + x}$

67. $\dfrac{1}{a^2 - x^2} = \dfrac{1}{(a+x)(a-x)}$

Review:

1. Evaluate: $\lim_{x \to -2} \dfrac{x^2 - 4}{x^3 + 8}$

$$\frac{\cancel{(x+2)}(x-2)}{\cancel{(x+2)}(x^2-2x+4)}$$

$$\frac{-2-2}{(-2)^2-2(-2)+4} = \frac{-4}{4+4+4} = -\frac{1}{3}$$

$\lim_{x \to 1} \dfrac{\sqrt{3} - \sqrt{x+2}}{1 - x} \cdot \dfrac{\sqrt{3} + \sqrt{x+2}}{\sqrt{3} + \sqrt{x+2}}$

$$\frac{3 - (x+2)}{(1-x)(\sqrt{3} + \sqrt{x+2})} =$$

$$\frac{3 - x - 2}{(1-x)(\sqrt{3} + \sqrt{x+2})} = \frac{\cancel{1-x}}{\cancel{(1-x)}(\sqrt{3} + \sqrt{x+2})}$$

$$\frac{1}{\sqrt{3} + \sqrt{x+2}} = \frac{1}{\sqrt{3} + \sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}$$

2. Solve for x: $\begin{vmatrix} x+4 & -2 \\ 7 & x-5 \end{vmatrix} = 0$

$(x+4)(x-5) - {}^-14 = 0$

$x^2 - x - 20 + 14 = 0$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = 3,\ x = -2$

3. Completely analyze the following function: $f(x) = \dfrac{6x^2 - 7x + 2}{4x^2 - 1} = \dfrac{(3x-2)\cancel{(2x-1)}}{(2x+1)\cancel{(2x-1)}}$

x-int: $(\frac{2}{3}, 0)$

y-int: $(0, -2)$

V.A: $x = -\frac{1}{2}$ $\qquad \lim_{x \to -\frac{1}{2}^-} f(x) = \infty$

$\lim_{x \to -\frac{1}{2}^+} f(x) = -\infty$

H.A.: $y = \frac{3}{2}$ $\qquad \lim_{x \to -\infty} f(x) = \frac{3}{2} = \lim_{x \to \infty} f(x)$

R.D.: $x = \frac{1}{2}$

hole: $(\frac{1}{2}, -.25)$

Lesson 3-4 Homework Key

21. $x(2x+1)\left[\dfrac{1}{x(2x+1)}\right] = \left[\dfrac{A}{x} + \dfrac{B}{(2x+1)}\right]x(2x+1)$

$1 = A(2x+1) + Bx$ $\quad 0 = 2A+B$

$1 = 2Ax + A + Bx$ $\quad 1 = A$

$1 = 2Ax + Bx + A$ $\quad 0 = 2+B$

$1 = x(2A+B) + A$ $\quad B = -2$

$$\boxed{\frac{1}{x(2x+1)} = \frac{1}{x} - \frac{2}{2x+1}}$$

23. $(x+2)(x-1)\left[\dfrac{3}{(x+2)(x-1)}\right] = \left[\dfrac{A}{x+2} + \dfrac{B}{x-1}\right](x+2)(x-1)$

$0 = A+B$

$3 = -A+2B$

$3 = 3B$

$B = 1$

$0 = A+1$

$A = -1$

$3 = A(x-1) + B(x+2)$

$3 = Ax - A + Bx + 2B$

$3 = Ax + Bx - A + 2B$

$3 = x(A+B) - A + 2B$

$$\boxed{\frac{3}{(x+2)(x-1)} = \frac{-1}{x+2} + \frac{1}{x-1}}$$

27. $x(x+2)(x-2)\left[\dfrac{x^2+12x+12}{x(x+2)(x-2)}\right] = \left[\dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{x-2}\right]x(x+2)(x-2)$

$x^2+12x+12 = A(x^2-4) + B(x)(x-2) + C(x)(x+2)$

$x^2+12x+12 = Ax^2 - 4A + Bx^2 - 2Bx + Cx^2 + 2Cx$

$x^2+12x+12 = Ax^2 + Bx^2 + Cx^2 - 2Bx + 2Cx - 4A$

$x^2+12x+12 = (A+B+C)x^2 + x(-2B+2C) - 4A$

$1 = A + B + C$

$12 = -2B + 2C$

$12 = -4A$

$A = -3$ ✓

$1 = -3 + B + C$

$(4 = B + C)2$

$12 = -2B + 2C$

$8 = 2B + 2C$

$20 = 4C$

$C = 5$ ✓

$1 = -3 + B + 5$

$B = -1$ ✓

$$\boxed{\frac{x^2+12x+12}{x(x+2)(x-2)} = \frac{-3}{x} - \frac{1}{x+2} + \frac{5}{x-2}}$$

29. $\cancel{(x-3)^2}\cdot\frac{3x}{\cancel{(x-3)^2}} = \left[\frac{A}{x-3} + \frac{B}{(x-3)^2}\right](x-3)^2$

$$3x = A(x-3) + B$$
$$3x = Ax - 3A + B$$
$$\boxed{\frac{3x}{(x-3)^2} = \frac{3}{x-3} + \frac{9}{(x-3)^2}}$$

$3 = A$
$0 = -3A + B$
$0 = -3(3) + B$
$B = 9$

31. $x^2(x+1)\left[\frac{4x^2+2x-1}{x^2(x+1)}\right] = \left[\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}\right]x^2(x+1)$

$$4x^2+2x-1 = Ax(x+1) + B(x+1) + Cx^2$$
$$4x^2+2x-1 = Ax^2 + Ax + Bx + B + Cx^2$$
$$4x^2+2x-1 = Ax^2 + Cx^2 + Ax + Bx + B$$
$$4x^2+2x-1 = (A+C)x^2 + (A+B)x + B$$
$$4x^2+2x-1$$

$A + C = 4$
$A + B = 2$
$B = -1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \quad \boxed{\frac{4x^2+2x-1}{x^2(x+1)} = \frac{3}{x} - \frac{1}{x^2} + \frac{1}{x+1}}$$

35. $(x-1)(x^2-2)\left[\frac{x}{(x^2-2)(x-1)}\right] = \left[\frac{A}{x-1} + \frac{Bx+C}{x^2-2}\right](x-1)(x^2-2)$

$$x = A(x^2-2) + (Bx+C)(x-1)$$
$$x = Ax^2 - 2A + Bx^2 - Bx + Cx - C$$
$$x = Ax^2 + Bx^2 - Bx + Cx - 2A - C$$
$$x = (A+B)x^2 + (-B+C)x - 2A - C$$

$A + B = 0$
$-B + C = 1$
$-2A - C = 0$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ -2 & 0 & -1 \end{bmatrix}\cdot\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \quad \boxed{\frac{x}{(x^2-2)(x-1)} = \frac{-1}{x-1} + \frac{x+2}{x^2-2}}$$

37.

$$\text{LCD}\left(\frac{2x^2+x+8}{(x^2+4)^2}\right)=\left(\frac{Ax+B}{x^2+4}+\frac{Cx+D}{(x^2+4)^2}\right)\text{LCD}$$

$$2x^2+x+8=(Ax+B)(x^2+4)+Cx+D$$

$$2x^2+x+8=Ax^3+Bx^2+4Ax+4B+Cx+D$$

$$2x^2+x+8=Ax^3+Bx^2+4Ax+Cx+4B+D$$

$0=A$ $\quad 4A+C=1$ $\quad 4B+D=8$

$2=B$ $\quad C=1$ $\quad 8+D=8$

$D=0$

$$\boxed{\frac{2x^2+x+8}{(x^2+4)^2}=\frac{2}{x^2+4}+\frac{x}{(x^2+4)^2}}$$

39. $\cancel{\text{LCD}}\cdot$

$$\left[\frac{x}{(4x^2+1)(2x+1)(2x-1)}\right]=\left[\frac{A}{2x+1}+\frac{B}{2x-1}+\frac{Cx+D}{4x^2+1}\right](2x+1)(2x-1)(4x^2+1)$$

$$x=A(2x-1)(4x^2+1)+B(2x+1)(4x^2+1)+(Cx+D)(4x^2-1)$$

$$x=A(8x^3-4x^2+2x-1)+B(8x^3+4x^2+2x+1)+4Cx^3+4Dx^2-Cx-D$$

$$x=8Ax^3-4Ax^2+2Ax-A+8Bx^3+4Bx^2+2Bx+B+4Cx^3+4Dx^2-Cx-D$$

$$x=8Ax^3+8Bx^3+4Cx^3-4Ax^2+4Bx^2+4Dx^2+2Ax+2Bx-Cx-A+B-D$$

$$x=(8A+8B+4C)x^3+(-4A+4B+4D)x^2+(2A+2B-C)x-A+B-D$$

$0=8A+8B+4C$

$0=-4A+4B+4D$

$1=2A+2B-C$

$0=-A+B-D$

$$\begin{bmatrix}8&8&4&0\\-4&4&0&4\\2&2&-1&0\\-1&1&0&-1\end{bmatrix}\cdot\begin{bmatrix}A\\B\\C\\D\end{bmatrix}=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

Thank you Matrices!

$$\begin{bmatrix}A\\B\\C\\D\end{bmatrix}=\begin{bmatrix}1/8\\1/8\\-1/2\\0\end{bmatrix}$$

$$\boxed{\frac{x}{(4x^2+1)(2x+1)(2x-1)}=\frac{1}{8(2x+1)}+\frac{1}{8(2x-1)}-\frac{x}{2(4x^2+1)}}$$

45. $2x-7+\frac{18x+19}{x^2+3x+2}=f(x)$

$$(x+2)(x+1)\left(\frac{18x+19}{(x+1)(x+2)}\right)=\left[\frac{A}{x+1}+\frac{B}{x+2}\right](x+2)(x+1)$$

$$18x+19=A(x+2)+B(x+1)$$
$$18x+19=Ax+2A+Bx+B$$
$$18x+19=Ax+Bx+2A+B$$
$$18x+19=x(A+B)+2A+B$$

$$18=A+B$$
$$-(19=2A+B)$$
$$-1=-A$$
$A=1 \quad B=17$

$$\boxed{f(x)=2x-7+\frac{1}{x+1}+\frac{17}{x+2}}$$

47. $f(x)=x+3+\frac{6x^2-8x+3}{\underbrace{(x-1)^3}}$

$$(x-1)^3\left[\frac{6x^2-8x+3}{(x-1)^3}\right]=\left[\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}\right](x-1)^3$$

$$6x^2-8x+3=A(x-1)^2+B(x-1)+C$$
$$6x^2-8x+3=A(x^2-2x+1)+Bx-B+C$$
$$6x^2-8x+3=Ax^2-2Ax+A+Bx-B+C$$
$$6x^2-8x+3=Ax^2-2Ax+Bx+A-B+C$$
$$6x^2-8x+3=Ax^2+(-2A+B)x+A-B+C$$

$A=6$
$-8=-2A+B$ $\quad -8=-2(6)+B$
$\quad B=4$
$3=A-B+C$
$3=6-4+C$
$C=1$

$$\boxed{\frac{x^4}{(x-1)^3}=\frac{6}{x-1}+\frac{4}{(x-1)^2}+\frac{1}{(x-1)^3}}$$

49. See next page

67. $$(a+x)(a-x)\left(\frac{1}{(a-x)(a+x)}\right)=\left[\frac{A}{a-x}+\frac{B}{a+x}\right](a+x)(a-x)$$

$$1=A(a+x)+B(a-x)$$
$$1=Aa+Ax+Ba-Bx$$
$$1=Ax-Bx+Aa+Ba$$
$$1=x(A-B)+Aa+Ba$$

$0=A-B \rightarrow A=B$
$1=Aa+Ba$ $\quad$ substi

$1=Aa+Aa$
$\frac{1}{2a}=\frac{2Aa}{2a}$
$\frac{1}{2a}=A$ $\quad A=B$ ✓
$\frac{1}{2a}=B$

$$\boxed{\frac{1}{(a-x)(a+x)}=\frac{1}{2a(a-x)}+\frac{1}{2a(a+x)}}$$

49.

$$\begin{array}{r} x \\ x^3+2x^2+x \overline{\smash{)}\,x^4+2x^3+4x^2+8x+2} \\ -(x^4+2x^3+x^2) \\ \hline 3x^2+8x+2 \end{array}$$

$$f(x) = x + \boxed{\frac{3x^2+8x+2}{x^3+2x^2+x}} \rightarrow \begin{array}{l} x(x^2+2x+1) \\ x(x+1)^2 \end{array}$$

$$x(x+1)^2\left[\frac{3x^2+8x+2}{x(x+1)^2}\right] = \left[\frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}\right]x(x+1)^2$$

$$3x^2+8x+2 = A(x+1)^2 + B(x)(x+1) + Cx$$

$$3x^2+8x+2 = A(x^2+2x+1) + Bx^2 + Bx + Cx$$

$$3x^2+8x+2 = \cancel{Ax^2} + \cancel{2Ax} + A + \cancel{Bx^2} + \cancel{Bx} + \cancel{Cx}$$

$$3x^2+8x+2 = Ax^2 + Bx^2 + 2Ax + Bx + Cx + A$$

$$3x^2+8x+2 = (A+B)x^2 + (2A+B+C)x + A$$

$$\begin{array}{l} 3 = 2 + B \\ B = 1 \end{array} \qquad \begin{cases} 3 = A + B \\ 8 = 2A + B + C \\ 2 = A \end{cases} \qquad \begin{array}{l} 8 = 2(2) + 1 + C \\ 8 = 5 + C \\ c = 3 \end{array}$$

$$\frac{3x^2+8x+2}{x(x+1)^2} = x + \frac{2}{x} + \frac{1}{x+1} + \frac{3}{(x+1)^2}$$